1

IIR Filter Design

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• Code : DSP9

1.1 Introduction :

Analog filters are designed using analog components like resistors (R), inductors (L) and capacitors (C). While digital filters are implemented using difference equation.

The digital filters described by differential equations can be implemented using software like 'C' or assembly language. We can easily change the algorithm; so we can easily change the filter characteristics according to our requirement.

Basically there are two types of filters as follows :

1. FIR (Finite impulse response) filter.
2. IIR (Infinite impulse response) filter.

We will study each type in detail; later in this chapter. Presently we will compare analog and digital filters by studying advantages and disadvantages of digital filters.

1.1.1 Advantages of Digital Filters :

1. Many input signals can be filtered by one digital filter without replacing the hardware.
2. Digital filters have characteristic like linear phase response. Such characteristic is not possible to obtain in case of analog filters.
3. The performance of digital filters, does not vary with environmental parameters. But the environmental parameters like temperature, humidity etc., change the values of components in case of analog filters. So it is required to calibrate analog filters periodically.
4. In case of digital filters; since the filtering is done with the help of digital computer, both filtered and unfiltered data can be saved for further use.
5. Unlike analog filters; the digital filters are portable.
6. From unit to unit the performance of digital filters is repeatable.
7. The digital filters are highly flexible.
8. Using VLSI technology; the hardware of digital filters can be reduced. Similarly the power consumption can be reduced.
9. Digital filters can be used at very low frequencies, for example in Biomedical applications.
10. In case of analog filters; maintenance is frequently required. But for digital filters it is not required.

1.1.2 Disadvantages of Digital Filters :

1. Speed limitation :

In case of digital filters, ADC and DAC are used. So the speed of digital filter depends on the conversion time of ADC and the settling time of DAC. Similarly the speed of operation of digital filter depends on the speed of digital processor. Thus the bandwidth of input signal processed is limited by ADC and DAC. In real time applications, the bandwidth of digital filter is much lower than analog filters.

2. Finite wordlength effect :

The accuracy of digital filter depends on the wordlength used to encode them in binary form. Wordlength should be long enough to obtain the required accuracy.

The digital filters are also affected by the ADC noise, resulting from the quantization of continuous signals. Similarly the accuracy of digital filters is also affected by the roundoff noise occurred during computation.

3. Long design and development time :

An initial design and development time for digital hardware is more than analog filters.

1.2 Filter Design Methods :

In order to design the digital IIR filter, analog IIR filter is designed first. Then analog filter is converted into the digital filter. Here you may ask a question, why to design digital filter from analog filter ?

The reasons are as follows :

(1) The procedure to design analog filter is readily available and it is highly advanced.

(2) When we design digital filter using analog filter then the implementation becomes simple.

The different methods use to design IIR filter are as follows :

(1) Approximation of derivatives. (2) Impulse invariance. (3) Bilinear transformation.
(4) Matched Z transform. (5) Least square filter design.

1.2.1 Approximation of Derivatives :

Consider an analog differentiator with transfer function $H_a(s)$. The function of analog differentiator is to take the derivative of analog input signal. Let $x(t)$ be the input signal applied to analog differentiator. Then its output can be written as,

$$y(t) = \frac{d}{dt} x(t) \qquad \text{...(1)}$$

Consider analog signal as shown in Fig. K-1(a).

Since the output is the differentiation of input, then from Fig. K-1(a) we can write,

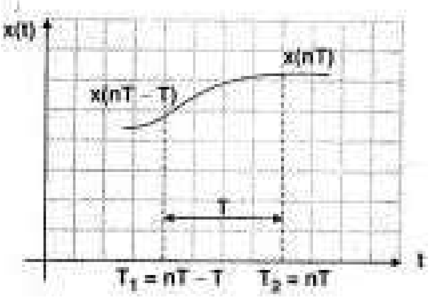

Fig. K-1(a) : Analog signal

$$y(nT) = \frac{x(nT) - x(nT-T)}{T}$$

$$\text{Thus } y(n) = \frac{x(n) - x(n-1)}{T} \quad ...(2)$$

Taking Z transform of both sides we get,

$$Y(Z) = \frac{X(Z) - Z^{-1}X(Z)}{T}$$

$$\therefore Y(Z) = X(Z)\left[\frac{1-Z^{-1}}{T}\right]$$

$$\therefore \frac{Y(Z)}{X(Z)} = H(Z) = \left[\frac{1-Z^{-1}}{T}\right] \quad ...(3)$$

Equation (3) gives the transfer function of digital filter.

Now we will obtain the transfer function of analog filter.

We have,

$$y(t) = \frac{d}{dt}x(t)$$

Taking Laplace transform we get,

$$Y(s) = sX(s)$$

$$\therefore \frac{Y(s)}{X(s)} = H(s) = s \quad ...(4)$$

Equation (4) gives the transfer function of analog filter.

Comparing Equations (3) and (4) we get,

$$s = \frac{1-Z^{-1}}{T} \quad ...(5)$$

Thus transfer function of digital filter is obtained by putting $s = \frac{1-Z^{-1}}{T}$ in the equation of the transfer function of digital filter.

That means,

$$H(Z) = H_a(s)\Big|_{s=\frac{1-Z^{-1}}{T}} \quad ...(6)$$

Mapping between s and Z plane :

We have,

$$s = \frac{1-Z^{-1}}{T}$$

Multiplying numerator and denominator by Z we get,

$$s = \frac{Z-1}{ZT}$$

$$\therefore \; sZT = Z - 1$$

$$\therefore \; sZT - Z = -1$$

$$\therefore \; -sZT + Z = 1$$

$$\therefore \; Z(1 - sT) = 1$$

$$\therefore \; Z = \frac{1}{1 - sT} \qquad \text{...(7)}$$

We know that 's' is the laplace operator and it is expressed as,

$$s = \sigma + j\,\Omega$$

Putting this value in Equation (7) we get,

$$Z = \frac{1}{1 - T(\sigma + j\,\Omega)}$$

$$\therefore \; Z = \frac{1}{1 - \sigma T - j\,\Omega T}$$

$$\therefore \; Z = \frac{1}{1 - \sigma T - j\,\Omega T} \times \frac{1 - \sigma T + j\,\Omega T}{1 - \sigma T + j\,\Omega T}$$

$$\therefore \; Z = \frac{1 - \sigma T + j\,\Omega T}{(1 - \sigma T)^2 + (\Omega T)^2}$$

$$\therefore \; Z = \frac{1 - \sigma T}{(1 - \sigma T)^2 + (\Omega T)^2} + j\,\frac{\Omega T}{(1 - \sigma T)^2 + (\Omega T)^2} \qquad \text{...(8)}$$

Putting $\sigma = 0$ in Equation (8) we get,

$$Z = \frac{1}{1 + (\Omega T)^2} + j\,\frac{\Omega T}{1 + (\Omega T)^2} \qquad \text{...(9)}$$

Now as Ω varies from $-\infty$ to $+\infty$, the corresponding locus of points in Z plane is a circle of radius $\frac{1}{2}$ and its centre at $Z = \frac{1}{2}$. This mapping is shown in Fig. K-1(b).

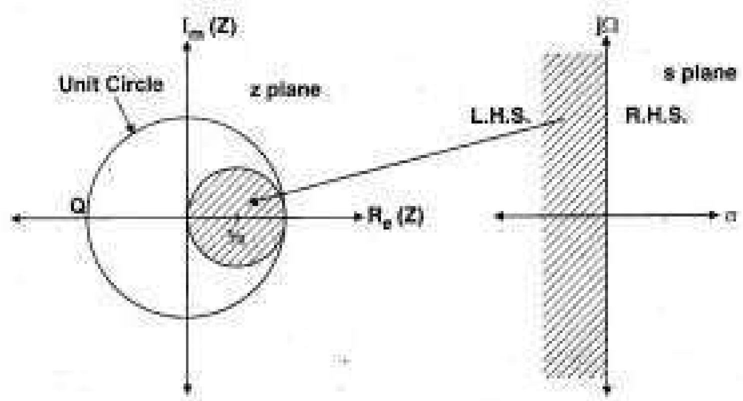

Fig. K-1(b) : Mapping between s and Z plane

This mapping shows that,

i) L.H.S. of s plane is mapped onto the points inside the circle in Z plane havin,. radius = $\frac{1}{2}$ and centre at $Z = \frac{1}{2}$.

ii) R.H.S. of s plane is mapped onto the points outside the circle in Z plane.

iii) The stable analog filter is converted into stable digital filter.

Limitation of approximation of derivatives method :

This method is suitable only for designing of low pass and bandpass IIR digital filters with relatively small resonant frequencies.

1.2.2 Impulse Invariance Method :

In this method, the design starts from the specifications of analog filter. Here w. have to replace analog filter by digital filter. This is achieved if impulse response of digital filter r.sembles the sampled version of impulse response of analog filter. If impulse response of both, analog and digital filter matches then, both filters perform in a similar manner.

Before studying this method we will list out the different notations, we are going to use,

$h(t)$ = Impulse response in time domain

$H_a(s)$ = Transfer function of analog filter; here 's' is laplace operator

$h(nT_s)$ = Sampled version of $h(t)$, obtained by replacing t by nT_s.

$H(Z)$ = Z transform of $h(nT_s)$. This is response of digital filter.

Ω = Analog frequency

ω = Digital frequency

Transformation of analog system function $H_a(s)$ to digital system function $H(Z)$:

Now let the system transfer function of analog filter be $H_a(s)$. We can express $H_a(s)$ in terms of partial fraction expansion. That means,

$$H_a(s) = \frac{A_1}{s-P_1} + \frac{A_2}{s-P_2} + \frac{A_3}{s-P_3} \ldots$$

$$\therefore \quad H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s-P_k} \qquad \ldots(1)$$

Here $A_k = A_1, A_2 \ldots A_N$ are the coefficients of partial fraction expansion,

and $P_k = P_1, P_2 \ldots P_N$ are the poles.

Here 's' is the laplace operator. So we can obtain impulse response of analog filter, $h(t)$ from $H_a(s)$ by taking inverse laplace of $H_a(s)$. So using standard relation of inverse laplace we get,

$$h(t) = \sum_{k=1}^{N} A_k e^{P_k t} \qquad ...(2)$$

Now unit impulse response for discrete structure is obtained by sampling $h(t)$. That means, $h(n)$ can be obtained from $h(t)$ by replacing 't' by $n T_s$ in Equation (2).

$$\therefore \quad h(n) = \sum_{k=1}^{N} A_k e^{P_k n T_s} \qquad ...(3)$$

Here T_s is the sampling time.

The system transfer function of digital filter is denoted by $H(Z)$. It is obtained by taking Z-transform of $h(n)$. According to the definition of Z-transform for causal system,

$$H(Z) = \sum_{n=0}^{\infty} h(n) Z^{-n} \qquad ...(4)$$

Putting Equation (3) in Equation (4) we get,

$$H(Z) = \sum_{n=0}^{\infty} \left[\sum_{k=1}^{N} A_k e^{P_k n T_s} \right] \cdot Z^{-n}$$

$$\therefore \quad H(Z) = \sum_{k=1}^{N} A_k \sum_{n=0}^{\infty} e^{P_k T_s n} \cdot Z^{-n}$$

$$\therefore \quad H(Z) = \sum_{k=1}^{N} A_k \sum_{n=0}^{\infty} \left(e^{P_k T_s} \cdot Z^{-1} \right)^n \qquad ...(5)$$

Using the standard summation formula,

$$\sum_{n=0}^{\infty} a^k = \frac{1}{1-a};$$

Equation (5) becomes,

$$H(Z) = \sum_{k=1}^{N} A_k \cdot \frac{1}{1 - e^{P_k T_s} Z^{-1}} \qquad ...(6)$$

This is the required transfer function of digital filter.

Thus comparing Equation (1) and Equation (6), we can say that the transfer function of digital filter is obtained from the transfer function of analog filter by doing the transformation.

$$\frac{1}{s - P_k} \rightarrow \frac{1}{1 - e^{P_k T_s} Z^{-1}} \quad ...(7)$$

Equation (7) shows, how the poles from analog domain are transferred into the digital domain. This transformation of poles is called as mapping of poles.

Relationship of s-plane to Z plane :

We know that the poles of analog filters are located at $s = P_k$. Now from Equation (7) we can say that the poles of digital filter, H (Z) are located at,

$$Z = e^{P_k T_s} \quad ...(8)$$

This equation indicates that the poles of analog filter at $s = P_k$ are transformed into the poles of digital filter at $Z = e^{P_k T_s}$. Thus the relationship between laplace ('s' domain) and Z domain is given by,

$$Z = e^{s T_s} \quad ...(9)$$

Here $s = P_k$ and T_s is the sampling time.

Now 's' is the laplace operator and it is expressed as,

$$s = \sigma + j\,\Omega \quad ...(10)$$

Here σ = Attenuation factor

and Ω = Analog frequency

We know the 'Z' can be expressed in polar form as,

$$Z = r\,e^{j\omega} \quad ...(11)$$

Here 'r' is magnitude and 'ω' is the digital frequency.

Putting Equations (10) and (11) in Equation (9) we get,

$$r\,e^{j\omega} = e^{(\sigma + j\Omega) T_s}$$

$$\therefore \quad r\,e^{j\omega} = e^{\sigma T_s} \cdot e^{j\Omega T_s} \quad ...(12)$$

Separating real and imaginary parts of Equation (12) we get,

$$r = e^{\sigma T_s} \quad ...(13)$$

$$\text{and} \quad e^{j\omega} = e^{j\Omega T_s}$$

$$\therefore \quad \omega = \Omega T_s \quad ...(14)$$

Now we will find the relationship between s plane and Z plane. Basically plot in 's'-domain means, σ is plotted on X-axis and $j\Omega$ is plotted on Y- axis. And Z-domain representation means real Z is plotted on X- axis and imaginary Z is plotted on Y-axis.

Now consider Equation (13), it is

$$r = e^{\sigma T_s}$$

We will discuss the following conditions :

(i) If $\sigma < 0$, then r is equal to reciprocal of 'e' raise to some constant. Thus range of r will be 0 to 1.

$$\therefore \sigma < 0 \Rightarrow 0 < r < 1 \quad ...(15)$$

Now $\sigma < 0$ means negative values of σ. That is L.H.S. of s plane. We know that 'r' is the radius of circle is Z plane.

So '$0 < r < 1$' indicates interior part of unit circle. Thus we can conclude that,

L.H.S. of 's' plane is mapped inside the unit circle.

(ii) If $\sigma = 0$ then $r = e^0 = 1$

$$\therefore \sigma = 0 \Rightarrow r = 1$$

Now $\sigma = 0$ indicates $j\Omega$ axis and $r = 1$ indicates unit circle. Thus,

$j\Omega$ axis in 's' plane is mapped on the unit circle.

(iii) If $\sigma > 0$ then, r is equal to 'e' raise to some constant. That means $r > 1$.

$$\therefore \sigma > 0 \Rightarrow r > 1$$

Now $\sigma > 0$ indicates R.H.S. of 's' plane and 'r' > 1 indicates exterior part of unit circle. Thus,

R.H.S. of 's' plane is mapped outside the unit circle.

Combining all conditions; this mapping is shown in Fig. K-2.

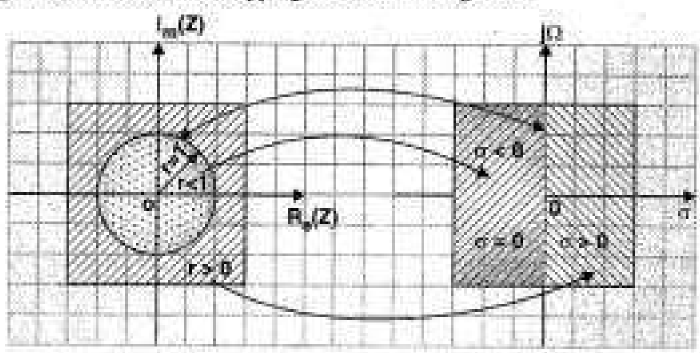

Fig. K-2 : Relationship of s plane to Z-plane

Disadvantages of impulse invariance method :

(1) We know that 'Ω' is analog frequency and its range is from $\frac{\pi}{T_s}$ to $-\frac{\pi}{T_s}$. While the digital frequency 'ω' varies from $-\pi$ to π. That means from $\frac{\pi}{T_s}$ to $-\frac{\pi}{T_s}$ 'ω' maps from $-\pi$ to π. Let k be any integer. Then, we can write the general range of Ω as $(k-1)\frac{\pi}{T_s}$ to $(k+1)\frac{\pi}{T_s}$; but

for this range also; 'ω' maps from $-\pi$ to π. Thus mapping from analog frequency 'Ω' to digital frequency 'ω' is many to one. This mapping is not one to one.

(2) Analog filters are not band limited so there will be aliasing due to the sampling process. Because of this aliasing, the frequency response of resulting digital filter will not be identical to the original frequency response of analog filter.

(3) The change in the value of sampling time (T_s) has no effect on the amount of aliasing.

Some standard formulae for transformation in impulse invariance method are as follows :

(i) $$\frac{1}{s-P_k} \longrightarrow \frac{1}{1-e^{P_k T_s} \cdot Z^{-1}}$$

(ii) $$\frac{s+a}{(s+a)^2+b^2} \longrightarrow \frac{1-e^{-aT_s}[\cos bT_s]Z^{-1}}{1-2e^{-aT_s}[\cos bT_s]Z^{-1}+e^{-2aT_s} \cdot Z^{-2}}$$

(iii) $$\frac{b}{(s+a)^2+b^2} \longrightarrow \frac{e^{-aT_s}[\sin bT_s]Z^{-1}}{1-2e^{-aT_s}[\cos bT_s]Z^{-1}+e^{-2aT_s} \cdot Z^{-2}}$$

Design steps for impulse invariance method :

Step I : Analog frequency transfer function H (s) will be given. If it is not given then, obtain expression of H (s) from the given specifications.

Step II : If required expand H (s) by using partial fraction expansion (PFE).

Step III : Obtain Z transform of each PFE term using impulse invariance transformation equation.

Step IV : Obtain H (Z), this is required digital IIR filter.

Solved Problems :

Prob. 1 : Find out H (Z) using impulse invariance method at 5 Hz sampling frequency from H (s) as given below :

$$H(s) = \frac{2}{(s+1)(s+2)}$$

Soln. :

Step I : Given analog transfer function is,

$$H(s) = \frac{2}{(s+1)(s+2)} \quad ...(1)$$

Step II : We will expand H (s) using partial fraction expansion as :

$$\therefore \quad H(s) = \frac{A_1}{(s+1)} + \frac{A_2}{(s+2)} \quad ...(2)$$

Thus poles are at $P_1 = -1$ and $P_2 = -2$.

Now values of A_1 and A_2 are calculated as follows :

$$A_1 = (s-P_1)H(s)\Big|_{s=P_1}$$

$$\therefore \quad A_1 = (s+1)\cdot\frac{2}{(s+1)(s+2)}\bigg|_{s=-1}$$

$$\therefore \quad A_1 = \frac{2}{-1+2} = 2$$

$$\text{and} \quad A_2 = (s-P_2)H(s)\bigg|_{s=P_2} = (s+2)\cdot\frac{2}{(s+1)(s+2)}\bigg|_{s=-2}$$

$$\therefore \quad A_2 = \frac{2}{-2+1} = -2$$

Putting values of A_1 and A_2 in Equation (2) we get,

$$H(s) = \frac{2}{(s+1)} - \frac{2}{(s+2)} \qquad ...(3)$$

Step III : Now we will obtain the Z-transform using impulse invariance transformation equation. It is,

$$\frac{1}{s-P_k} \longrightarrow \frac{1}{1-e^{P_k T_s}\cdot Z^{-1}} \qquad ...(4)$$

Here T_s = Sampling time. Now given sampling frequency is F_s = 5 Hz.

$$\therefore \quad T_s = \frac{1}{F_s} = \frac{1}{5} = 0.2 \text{ sec.}$$

we have poles at $P_1 = -1$ and $P_2 = -2$

So using Equation (4) we get,

$$\frac{1}{s+1} \longrightarrow \frac{1}{1-e^{-1(0.2)}\cdot Z^{-1}} = \frac{1}{1-e^{-0.2}\cdot Z^{-1}} \qquad ...(5)$$

$$\text{and} \quad \frac{1}{s+2} \longrightarrow \frac{1}{1-e^{-2(0.2)}\cdot Z^{-1}} = \frac{1}{1-e^{-0.4} Z^{-1}} \qquad ...(6)$$

Step IV : The transfer function of digital filter is given by,

$$H(Z) = \sum_{k=1}^{N} \frac{A_k}{1-e^{P_k T_s}\cdot Z^{-1}}$$

In this case we get,

$$H(Z) = \frac{A_1}{1-e^{P_1 T_s}\cdot Z^{-1}} + \frac{A_2}{1-e^{P_2 T_s}\cdot Z^{-1}} \qquad ...(7)$$

Using Equations (5) and (6) we get,

$$H(Z) = \frac{2}{1-e^{-0.2}\cdot Z^{-1}} - \frac{2}{1-e^{-0.4}\cdot Z^{-1}}$$

$$\therefore \quad H(Z) = \frac{2}{1-0.818\,Z^{-1}} - \frac{2}{1-0.67\,Z^{-1}}$$

To convert each term into positive powers of Z; multiplying numerator and denominator of each term by Z we get,

$$H(Z) = \frac{2Z}{Z-0.818} - \frac{2Z}{Z-0.67}$$

$$\therefore \quad H(Z) = \frac{2Z(Z-0.67) - 2Z(Z-0.818)}{(Z-0.818)(Z-0.67)}$$

$$\therefore \quad H(Z) = \frac{2Z^2 - 1.34Z - 2Z^2 + 1.636Z}{Z^2 - 0.67Z - 0.818Z + 0.54}$$

$$\therefore \quad H(Z) = \frac{0.29Z}{Z^2 - 1.488Z + 0.54}$$

$$\therefore \quad H(Z) = \frac{0.29Z^{-1}}{1 - 1.488Z^{-1} + 0.54Z^{-2}}$$

This is the required transfer function for digital IIR filter.

Prob. 2 : Determine H (Z) using impulse invariance method for the system function,

$$H(s) = \frac{1}{(s+0.5)(s^2+0.5s+2)}$$

Soln. :

Step I : The given transfer function is,

$$H(s) = \frac{1}{(s+0.5)(s^2+0.5s+2)} \qquad ...(1)$$

Step II : In the partial fraction expansion form H (s) can be written as,

$$H(s) = \frac{1}{(s+0.5)(s^2+0.5s+2)} = \frac{A}{(s+0.5)} + \frac{Bs+C}{(s^2+0.5s+2)} \qquad ...(2)$$

Let us obtain the values of A, B and C.

$$\frac{1}{(s+0.5)(s^2+0.5s+2)} = \frac{A(s^2+0.5s+2)+(Bs+C)(s+0.5)}{(s+0.5)(s^2+0.5s+2)}$$

$$\therefore \quad A(s^2+0.5s+2)+(Bs+C)(s+0.5) = 1$$

$$\therefore \quad As^2 + A\,0.5s + 2A + Bs^2 + B\,0.5s + C.s + 0.5C = 1$$

$$s^2(A+B) + s(0.5A + 0.5B + C) + (2A + 0.5C) = 1 \qquad ...(3)$$

Now s^2 term is absent in R.H.S.

$$\therefore \quad A + B = 0 \qquad ...(4)$$

Similarly 's' term is absent in R.H.S.

$$\therefore \quad 0.5A + 0.5B + C = 0 \qquad ...(5)$$

$$\text{And} \quad 2A + 0.5C = 1 \qquad ...(6)$$

Now we will solve Equations (4), (5) and (6) to obtain the values of A, B and C.

From Equation (4),

$$B = -A$$

Putting this value in Equation (5) we get,

$$0.5\,A - 0.5\,A + C = 0$$

$$\therefore \ C = 0$$

From Equation (6), $2A + 0 = 1$

$$\therefore \ A = 0.5$$

Since $B = -A$ we get,

$$\therefore \ B = -0.5$$

Putting these values in Equation (2) we get,

$$H(s) = \frac{0.5}{s+0.5} - \frac{0.5\,s}{s^2+0.5\,s+2} \quad ...(7)$$

To use the standard transformation formulae we will convert the second term on R.H.S. in the form $\frac{s+a}{(s+a)^2+b^2}$ and $\frac{b}{(s+a)^2+b^2}$

Consider the term $s^2+0.5\,s+2$. It can be expressed as,

$$s^2+0.5\,s+2 = (s^2+0.5\,s+0.0625)+(1.9375)$$

$$\therefore \quad s^2+0.5\,s+2 = (s+0.25)^2+(1.39)^2$$

Putting this value in Equation (7) we get,

$$H(s) = \frac{0.5}{s+0.5} - \frac{0.5\,s}{(s+0.25)^2+(1.39)^2}$$

$$\therefore \quad H(s) = \frac{0.5}{s+0.5} - 0.5\left[\frac{s+0.25-0.25}{(s+0.25)^2+(1.39)^2}\right]$$

$$\therefore \quad H(s) = \frac{0.5}{s+0.5} - 0.5\left[\frac{s+0.25}{(s+0.25)^2}\right] - 0.5\left[\frac{-0.25}{(s+0.25)^2+(1.39)^2}\right]$$

Now we want the numerator of third term equal to 1.39. It is arranged as follows,

$$H(s) = \frac{0.5}{s+0.5} - 0.5\left[\frac{s+0.25}{(s+0.25)^2}\right] + \frac{0.5\times0.25}{1.39}\left[\frac{1.39}{(s+0.25)^2+(1.39)^2}\right]$$

$$\therefore \quad H(s) = \frac{0.5}{s+0.5} - 0.5\left[\frac{s+0.25}{(s+0.25)^2}\right] + 0.089\left[\frac{1.39}{(s+0.25)^2+(1.39)^2}\right] \quad ...(8)$$

Now recall the standard transformation formulae.

(i) $$\frac{1}{s-P_k} \longrightarrow \frac{1}{1-e^{P_k T_s} \cdot Z^{-1}}$$

(ii) $$\frac{s+a}{(s+a)^2+b^2} \longrightarrow \frac{1-e^{-aT_s}[\cos bT_s]Z^{-1}}{1-2e^{-aT_s}[\cos bT_s]Z^{-1}+e^{-2aT_s} \cdot Z^{-2}}$$

(iii) $$\frac{b}{(s+a)^2+b^2} \longrightarrow \frac{e^{-aT_s}[\sin bT_s]Z^{-1}}{1-2e^{-aT_s}[\cos bT_s]Z^{-1}+e^{-2aT_s} \cdot Z^{-2}}$$

Step III : Using these formulae, equation of H (Z) can be obtained from Equation (8) as follows :

$$H(Z) = \frac{0.5}{1-e^{0.5\,T_s} \cdot Z^{-1}} - 0.5\left[\frac{1-e^{-0.25\,T_s}[\cos 1.39\,T_s]Z^{-1}}{1-2e^{-0.25\,T_s}[\cos 1.39\,T_s]Z^{-1}+e^{-0.5\,T_s} \cdot Z^{-2}}\right]$$
$$+\,0.089\left[\frac{e^{-0.25\,T_s}(\sin 1.39\,T_s)Z^{-1}}{1-2e^{-0.25\,T_s}(\cos 1.39\,T_s)Z^{-1}+e^{-0.5\,T_s} \cdot Z^{-2}}\right]$$

This is the required transfer function. Note that the value of T_s is not given in the problem; so T_s is kept as it is.

1.2.3 Bilinear Transformation Method :

In case of impulse invariance method, we have studied that the mapping is many to one. So this method is not suitable to design high-pass filter and band reject filter.

In case of bilinear transformation; the mapping is one to one from s domain to the Z domain. So there is no aliasing effect. The limitations of impulse invariance method are overcome by using BLT method.

Consider an analog integrator as shown in Fig. K-2(A)

Fig. K-2(A)

The response of analog integrator is,

$$H_a(s) = \frac{1}{s} \qquad ...(1)$$

Input in laplace domain is X (s) and output in laplace domain is Y (s).

For time period T; the difference in output is given by,

$$Y(t_2) - Y(t_1) = \int_{t_1}^{t} x(nT_s)\,dT \qquad ...(2)$$

Consider input signal as shown in Fig. K-2(B).

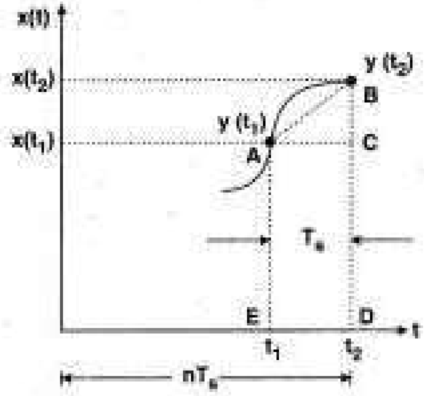

Fig. K-2(B)

Here we have assumed two input positions $x(t_1)$ and $x(t_2)$. Corresponding output is denoted by $Y(t_1)$ and $Y(t_2)$ respectively. Now area under the curve is addition of area of triangle ABC and area of rectangle ACDE.

$$\therefore \quad Y(t_2) - Y(t_1) = \frac{1}{2}(t_2 - t_1)\Big[x(t_2) - x(t_1)\Big] + x(t_1)(t_2 - t_1) \qquad ...(3)$$

As shown in Fig. K-2(B), time period $t_2 = nT_S$. Thus time period t_1 is $nT_s - T_s$. That means $t_2 - t_1 = T_s$

Putting these values in Equation (3) we get,

$$Y(nT_s) - Y(nT_s - T_s) = \frac{1}{2}T\Big[x(nT_s) - x(nT_s - T)\Big] + x(nT_s - T)T_s$$

$$Y(nT_s) - Y(nT_s - T_s) = \frac{1}{2}T_s x(nT) - \frac{1}{2}T_s x(nT_s - T_s) + T_s x(nT_s - T_s)$$

$$\therefore \quad Y(nT_s) - Y(nT_S - T_s) = \frac{1}{2}T_s x(nT_s) + \frac{1}{2}T_s x(nT_s - T_s)$$

$$\therefore \quad Y(nT_s) - Y(nT_s) - T_s = \frac{T_s}{2}\Big[x(nT_s) + x(nT_s - T_s)\Big] \qquad ...(4)$$

Taking Z transform of both sides we get,

$$Y(Z) - Z^{-1}Y(Z) = \frac{T_s}{2}\Big[X(Z) + Z^{-1}X(Z)\Big]$$

$$\therefore Y(Z)\Big[1 - Z^{-1}\Big] = \frac{T_s}{2}\Big[X(Z)(1 + Z^{-1})\Big]$$

$$\therefore \quad \frac{Y(Z)}{X(Z)} = H(Z) = \frac{T_s}{2}\frac{(1+Z^{-1})}{(1-Z^{-1})} \qquad ...(5)$$

Now we have transfer function of analog filter,

$$H_a(s) = \frac{1}{s} \qquad ...(6)$$

Equating (5) and (6) we get,

$$\frac{1}{s} = \frac{T_s}{2}\frac{(1+Z^{-1})}{(1-Z^{-1})} \qquad ...(7)$$

Thus relationship between s plane and Z-plane is given by,

$$s = \frac{2}{T_s}\left(\frac{Z-1}{Z+1}\right) \qquad ...(8)$$

Here T_s is the sampling time.

We know that 's' is the laplace operator and it can be expressed as,

$$s = \sigma + j\Omega \qquad ...(9)$$

Now the equation of Z in polar form is,

$$Z = r e^{j\omega} \qquad ...(10)$$

Putting Equations (9) and (10) in Equation (8) and multiplying numerator and denominator by $(re^{-j\omega}+1)$ we get,

$$\sigma + j\Omega = \frac{2}{T_s}\left[\frac{re^{j\omega}-1}{re^{j\omega}+1}\right] \times \frac{re^{-j\omega}+1}{re^{-j\omega}+1}$$

$$\sigma + j\Omega = \frac{2}{T_s}\left[\frac{r^2 \cdot e^{j\omega} \cdot e^{-j\omega} + re^{j\omega} - re^{-j\omega} - 1}{r^2 e^{j\omega} \cdot e^{-j\omega} + re^{j\omega} + re^{-j\omega} + 1}\right] \qquad ...(11)$$

But $e^{j\omega} \cdot e^{-j\omega} = e^0 = 1$

$$\therefore \quad \sigma + j\Omega = \frac{2}{T_s}\left[\frac{r^2 + re^{j\omega} - re^{-j\omega} - 1}{r^2 + re^{j\omega} + re^{-j\omega} + 1}\right] \qquad ...(12)$$

Now we have, $\frac{e^{j\omega} - e^{-j\omega}}{2j} = \sin\omega$ and $\frac{e^{j\omega} + e^{-j\omega}}{2} = \cos\omega$.

We will rearrange Equation (12) as follows :

$$\sigma + j\Omega = \frac{2}{T_s}\left[\frac{r^2 - 1 + j2r\left(\frac{e^{j\omega} - e^{-j\omega}}{2j}\right)}{r^2 + 2r\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right) + 1}\right]$$

$$\therefore \quad \sigma + j\Omega = \frac{2}{T_s}\left[\frac{r^2 - 1 + j2r\sin\omega}{r^2 + 2r\cos\omega + 1}\right]$$

$$\therefore \quad \sigma + j\Omega = \frac{2}{T_s}\left[\frac{r^2-1}{r^2+2r\cos\omega+1} + j\frac{2r\sin\omega}{r^2+2r\cos\omega+1}\right] \qquad ...(13)$$

Equating real and imaginary parts of Equation (13) we get,

$$\sigma = \frac{2}{T_s} \times \frac{r^2-1}{r^2+2r\cos\omega+1} \qquad ...(14)$$

$$\text{and} \quad \Omega = \frac{2}{T_s} \times \frac{2r\sin\omega}{r^2+2r\cos\omega+1} \qquad ...(15)$$

Now we will discuss the following conditions related to Equation (14).

(i) **When $r < 1$ then $\sigma < 0$**

Here $r < 1$, means interior part of circle having unit circle and $\sigma < 0$, means σ is negative which is L.H.S. of s-plane. So this condition indicates that L.H.S. of s plane maps inside the unit circle.

(ii) **When $r = 1$ then $\sigma = 0$**

Now $r = 1$ means unit circle and $\sigma = 0$ means $j\Omega$ axis. Thus this condition indicates that the $j\Omega$ axis maps on the unit circle.

(iii) **When $r > 1$ then $\sigma > 0$**

Here $r > 1$, means exterior part of unit circle and $\sigma > 0$ indicates that σ is positive means R.H.S. of s-plane. So this condition indicates that R.H.S. of s-plane maps outside the unit circle.

This mapping is similar to the mapping in impulse invariance method, as shown in Fig. K-2. But in impulse invariance method mapping is valid only for poles ; while in bilinear transformation, mapping is valid for poles as well as zeros.

How stable analog filter is converted into stable digital filter ?

Analog filter is stable if the poles lie on the L.H.S. of s-plane. While the digital filter is stable if the poles are inside the unit circle in the Z-domain. Now condition (i) indicates that L.H.S. of s-plane maps inside the unit circle. Thus stable analog filter is converted into stable digital filter.

Frequency warping concept :

Here we will obtain the relationship of $j\Omega$ axis in s-plane to the unit circle in the Z-plane ($r = 1$).

Recall Equation (15).

$$\Omega = \frac{2}{T_s} \times \frac{2r\sin\omega}{r^2+2r\cos\omega+1}$$

For the unit circle, $r = 1$. Thus putting $r = 1$ in the equation of Ω we get,

$$\Omega = \frac{2}{T_s} \times \frac{2\sin\omega}{1+2\cos\omega+1}$$

$$\therefore \quad \Omega = \frac{2}{T_s} \times \frac{2\sin\omega}{2+2\cos\omega}$$

$$\therefore \quad \Omega = \frac{2}{T_s} \times \frac{\sin \omega}{1 + \cos \omega} \qquad ...(16)$$

We have the trignometric identities,

$$\sin \omega = 2 \sin \frac{\omega}{2} \cdot \cos \frac{\omega}{2} \text{ and } 2 \cos^2 \frac{\omega}{2} = 1 + \cos \omega.$$

Thus Equation (16) becomes,

$$\Omega = \frac{2}{T_s} \times \frac{2 \sin \frac{\omega}{2} \cos \frac{\omega}{2}}{2 \cos^2 \frac{\omega}{2}}$$

$$\therefore \quad \Omega = \frac{2}{T_s} \times \frac{2 \sin \frac{\omega}{2}}{2 \cos \omega / 2}$$

$$\therefore \quad \Omega = \frac{2}{T_s} \tan \frac{\omega}{2}$$

$$\therefore \quad \omega = 2 \tan^{-1} \left(\frac{\Omega T_s}{2} \right) \qquad ...(17)$$

Now for different values of ΩT_s; the graph of ΩT_s versus ω is as shown in Fig. K-3.

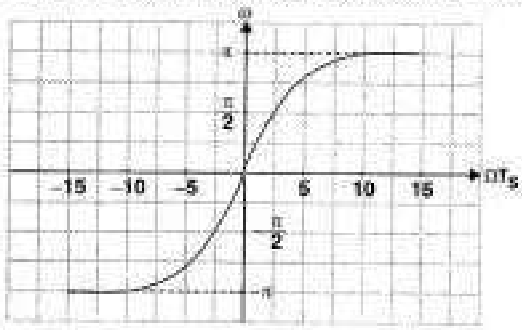

Fig. K-3 : Mapping between ω and Ω

Observations :

1. The entire range in Ω is mapped only once into the range $-\pi \leq \omega \leq \pi$.
2. The mapping is one to one.
3. The mapping is highly non-linear.

Now because of the non-linearity of tangent function $2 \tan^{-1} \left(\frac{\Omega T_s}{2} \right)$; there exists frequency warping or frequency compression.

What is frequency warping ?

Because of the non-linear mapping ; the amplitude response of digital IIR filter is expanded at lower frequencies and compressed at higher frequencies in comparison to the analog filter. This effect is called as frequency warping.

Prewarping procedure :

We have discussed the warping effect. Because of this effect, there is non-linear compression of Ω to ω values. To compensate this effect ; prewarping or prescaling procedure is used. This procedure is as follows :

(i) The 'Ω' scale is changed to prewarpped scale denoted by Ω^* and $\Omega^* = \frac{2}{T_s} \tan\left(\frac{\omega T_s}{2}\right)$

(ii) Then analog filter transfer function H (s) is obtained using values of Ω^*.

(iii) By applying the bilinear transformation, the desired digital frequency response H (Z) is obtained.

Advantages of bilinear transformation method :

1. There is one to one transformation from the s-domain to the Z- domain.
2. The mapping is one to one.
3. There is no aliasing effect.
4. Stable analog filter is transformed into the stable digital filter.

Disadvantage of bilinear transformation method :

The mapping is non-linear and because of this; frequency warping effect takes place.

Comparison between Impulse invariance and Bilinear transformation method :

Sr. No.	Impulse invariance method	Bilinear transformation method
1.	Poles are transferred by using the equation, $\frac{1}{s-P_k} \to \frac{1}{1-e^{P_k T_s} \cdot Z^{-1}}$	Poles are transferred by using the equation, $s = \frac{2}{T_s}\left[\frac{Z-1}{Z+1}\right]$
2.	Mapping is many to one.	Mapping is one to one.
3.	Aliasing effect is present.	Aliasing effect is not present.
4.	It is not suitable to design high-pass filter and band reject filter.	High pass filter and band reject filter can be designed.
5.	Only poles of the system can be mapped.	Poles as well as zeros can be mapped.
6.	No frequency warping effect.	Frequency warping effect is present.

Solved Problems :

Prob. 1 : An analog filter has the following transfer function $H(S) = \frac{1}{s+1}$. Using bilinear transformation technique, determine the transfer function of digital filter H (Z) and also write the difference equation of digital filter.

Soln. : The given transfer function is,

$$H(S) = \frac{1}{s+1} \qquad ...(1)$$

In bilinear transformation H(Z) is obtained by putting,

$$s = \frac{2}{T_s}\left[\frac{Z-1}{Z+1}\right]$$

Here T_s is the sampling time; which is not given. So assume $T_s = 1$ sec,

$$\therefore \quad s = 2\left[\frac{Z-1}{Z+1}\right] \qquad ...(2)$$

Putting this value in Equation (1),

$$H(Z) = \frac{1}{1+2\left(\frac{Z-1}{Z+1}\right)} = \frac{1}{\frac{(Z+1)+2(Z-1)}{Z+1}}$$

$$\therefore \quad H(Z) = \frac{Z+1}{Z+1+2Z-2}$$

$$\therefore \quad H(Z) = \frac{Z+1}{3Z-1} \qquad ...(3)$$

This is the transfer function of digital filter. Now we have,

$$H(Z) = \frac{Y(Z)}{X(Z)} = \frac{Z+1}{3Z-1}$$

Converting into negative power of Z we get,

$$\frac{Y(Z)}{X(Z)} = \frac{1+Z^{-1}}{3-Z^{-1}}$$

$$\therefore \quad Y(Z)(3-Z^{-1}) = X(Z)(1+Z^{-1})$$

$$\therefore \quad 3Y(Z)-Z^{-1}\cdot Y(Z) = X(Z)+Z^{-1}X(Z)$$

Taking IZT of both sides,

$$3y(n)-y(n-1) = x(n)+x(n-1)$$

$$\therefore \quad 3y(n) = x(n)+x(n-1)+y(n-1)$$

$$\therefore \quad y(n) = \frac{1}{3}x(n)+\frac{1}{3}x(n-1)+\frac{1}{3}y(n-1)$$

This is the difference equation of digital filter.

Prob. 2 : The transfer function of analog filter is :

$$H(s) = \frac{3}{(s+2)(s+3)} \text{ with } T_s = 0.1 \text{ sec.}$$

Design the digital IIR filter using BLT.

Soln. : The given transfer function is,

$$H(s) = \frac{3}{(s+2)(s+3)} \qquad ...(1)$$

...(2)

...(3)

(1)

In bilinear transformation, H (Z) is obtained by putting :

$$s = \frac{2}{T_s}\left[\frac{Z-1}{Z+1}\right] \quad ...(2)$$

Here T_s = sampling time = 0.1 sec.

$$s = \frac{2}{0.1}\left[\frac{Z-1}{Z+1}\right] = 20\left[\frac{Z-1}{Z+1}\right]$$

Putting this value in Equation (1) we get,

$$H(Z) = \frac{3}{\left[20\left(\frac{Z-1}{Z+1}\right)+2\right]\left[20\left(\frac{Z-1}{Z+1}\right)+3\right]}$$

$$\therefore \quad H(Z) = \frac{3}{\left[\frac{20Z-20}{Z+1}+2\right]\left[\frac{20Z-20}{Z+1}+3\right]}$$

$$\therefore \quad H(Z) = \frac{3(Z+1)(Z+1)}{(20Z-20+2Z+2)(20Z-20+3Z+3)}$$

$$\therefore \quad H(Z) = \frac{3(Z+1)^2}{(22Z-18)(23Z-17)}$$

$$\therefore \quad H(Z) = \frac{3(Z^2+2Z+1)}{506Z^2-374Z-414Z+306}$$

$$\therefore \quad H(Z) = \frac{3(Z^2+2Z+1)}{506Z^2-788Z+306} = \frac{Z^2+2Z+1}{168.67Z^2-262.67Z+102}$$

$$= \frac{1+2Z^{-1}+Z^{-2}}{168.67-262.67Z^{-1}+102Z^{-2}}$$

This is the required transfer function for digital IIR filter.

Prob. 3 : Design a single pole low pass digital filter with a 3 dB bandwidth of 0.2 π by use of bilinear transformation applied to the analog filter.

$$H(s) = \frac{\Omega_c}{s+\Omega_c}$$, where Ω_c is the 3 dB bandwidth of analog filter.

Soln. : The given transfer function of analog filter is,

$$H(s) = \frac{\Omega_c}{s+\Omega_c} \quad ...(1)$$

The cut-off frequency of digital filter (ω_c) is given. It is,

$$\omega_c = 0.2\pi \quad ...(2)$$

We know that in BLT, Ω_c and ω_c are related as,

$$\Omega_c = \frac{2}{T_s}\tan\left[\frac{\omega_c}{2}\right] = \frac{2}{T_s}\tan\left[\frac{0.2\,\pi}{2}\right] = \frac{2}{T_s}\tan[0.1\,\pi]$$

$$\therefore \quad \Omega_c = \frac{0.65}{T_s} \qquad \text{...(3)}$$

Putting this value in Equation (1) we get,

$$H(s) = \frac{\frac{0.65}{T_s}}{s + \frac{0.65}{T_s}} \qquad \text{...(4)}$$

Thus we have obtained the transfer function of analog filter in the appropriate form.

Now to obtain H (Z) we have to put,

$$s = \frac{2}{T_s}\left[\frac{Z-1}{Z+1}\right] \text{ in Equation (4)}$$

$$\therefore \quad H(Z) = \frac{0.65/T_s}{\frac{2}{T_s}\left[\frac{Z-1}{Z+1}\right] + \frac{0.65}{T_s}}$$

Multiplying numerator and denominator by T_s we get,

$$H(Z) = \frac{0.65}{2\left[\frac{Z-1}{Z+1}\right] + 0.65} = \frac{0.65}{\frac{2Z-2}{Z+1} + 0.65}$$

$$\therefore \quad H(Z) = \frac{0.65(Z+1)}{2Z - 2 + 0.65\,Z + 0.65} = \frac{0.65(Z+1)}{2.65\,Z - 1.35}$$

Dividing numerator and denominator by 2.65 we get,

$$H(Z) = \frac{0.245(Z+1)}{Z - 0.509} \qquad \therefore \; H(Z) = \frac{0.245\,[1 + Z^{-1}]}{1 - 0.509\,Z^{-1}} \qquad \text{...(5)}$$

This is the required transfer function for digital filter. Note that there is a single pole at $P_1 = 0.509$.

Now in the problem frequency of digital filter is given which is $\omega_c = 0.2\,\pi$ and bandwidth is 3 dB. We will check the frequency response. The frequency response of digital filter is obtained by putting $Z = e^{j\omega}$ in the equation of H (Z). Thus we get,

$$H(\omega) = H(Z)\Big|_{Z = e^{j\omega}}$$

$$\therefore \quad H(\omega) = \frac{0.245\,(e^{j\omega} + 1)}{e^{j\omega} - 0.509} \qquad \text{...(6)}$$

If $\omega = 0$ then Equation (6) becomes,

$$H(0) = \frac{0.245(e^{0}+1)}{e^{0}-0.509} = \frac{0.49}{0.49} = 1$$

Thus at $\omega = 0$; the magnitude $|H(\omega)| = 1$.

Now at $\omega = 0.2\pi$ we get,

$$H(\omega) = \frac{0.245(e^{j0.2\pi}+1)}{e^{j0.2\pi}-0.509} = \frac{0.245[\cos(0.2\pi)+j\sin(0.2\pi)+1]}{\cos(0.2\pi)+j\sin(0.2\pi)-0.509}$$

$$\therefore \quad H(\omega) = \frac{0.245[0.809+j\,0.587+1]}{0.809+j\,0.587-0.509}$$

$$\therefore \quad H(\omega) = \frac{0.245(1.809+j\,0.589)}{0.3+j\,0.587}$$

$$\therefore \quad |H(\omega)| = 0.707 = 3\text{ dB}$$

Thus the desired response is obtained.

Prob. 4 : The system transfer function of analog filter is given by,

$$H(s) = \frac{s+0.1}{(s+0.1)^2+16}$$

Obtain the system transfer function of digital filter using BLT which is resonant at $\omega_r = \frac{\pi}{2}$

Soln. : The given transfer function is,

$$H(s) = \frac{s+0.1}{(s+0.1)^2+16}$$

$$\therefore \quad H(s) = \frac{s+0.1}{(s+0.1)^2+(4)^2} \qquad ...(1)$$

From Equation (1) we can say that $\Omega = 4$.

The value of ω_r is given as $\omega_r = \frac{\pi}{2}$

Now we will find out the value of sampling time (T_s) using the relation,

$$\Omega = \frac{2}{T_s}\tan\left(\frac{\omega}{2}\right)$$

$$\therefore \quad 4 = \frac{2}{T_s}\tan\left(\frac{\pi}{4}\right)$$

$$\therefore \quad T_s = \frac{2}{4}\tan\left(\frac{\pi}{4}\right)$$

$$\therefore \quad T_s = 0.5\text{ sec.} \qquad ...(2)$$

Using bilinear transformation H (Z) can be obtained by putting, $s = \frac{2}{T_s}\left(\frac{Z-1}{Z+1}\right)$ in the equation of H (s).

$$\therefore \quad H(Z) = \frac{\frac{2}{0.5}\left(\frac{Z-1}{Z+1}\right)+0.1}{\left[\frac{2}{0.5}\left(\frac{Z-1}{Z+1}\right)+0.1\right]^2+16}$$

$$\therefore \quad H(Z) = \frac{4\left(\frac{Z-1}{Z+1}\right)+0.1}{\left[4\left(\frac{Z-1}{Z+1}\right)+0.1\right]^2+16} = \frac{\frac{4Z-4}{Z+1}+0.1}{\left[\frac{4Z-4}{Z+1}+0.1\right]^2+16}$$

$$\therefore \quad H(Z) = \frac{\frac{4Z-4}{Z+1}+0.1}{\left[\frac{4Z-4+0.1Z+0.1}{Z+1}\right]^2+16} = \frac{\frac{4Z-4+0.1Z+0.1}{Z+1}}{\left[\frac{4.1Z-3.9}{Z+1}\right]^2+16}$$

$$\therefore \quad H(Z) = \frac{\frac{4.1Z-3.9}{Z+1}}{\frac{(16.81Z^2-31.98Z+15.21)}{(Z+1)^2}+16}$$

$$\therefore \quad H(Z) = \frac{\frac{(4.1Z-3.9)}{Z+1}\times(Z+1)^2}{16.81Z^2-31.98Z+15.21+16(Z+1)^2}$$

$$\therefore \quad H(Z) = \frac{(4.1Z-3.9)(Z+1)}{16.81Z^2-31.98Z+15.21+16Z^2+32Z+16}$$

$$\therefore \quad H(Z) = \frac{4.1Z^2+4.1Z-3.9Z-3.9}{32.81Z^2+0.02Z+31.21}$$

$$\therefore \quad H(Z) = \frac{4.1Z^2+0.2Z-3.9}{32.81Z^2+0.02Z+31.21}$$

$$\therefore \quad H(Z) = \frac{4.1+0.2Z^{-1}-3.9Z^{-2}}{32.81+0.02Z^{-1}+31.2/Z^{-2}}$$

This is the required transfer function for digital IIR filter.

in the

Prob. 5 : The analog transfer function of low pass filter is, $H(s) = \frac{1}{s+2}$ and its bandwidth is 1 rad/sec.

Design the digital filter using BLT method whose cut-off frequency is 20 π and sampling time is 0.0167 sec.; by considering the warping effect.

Soln. : Given analog transfer function is,

$$H(s) = \frac{1}{s+2} \quad ...(1)$$

For the prewarping procedure we have,

$$\Omega_p^* = \frac{2}{T_s} \tan\left(\frac{\omega T_s}{2}\right)$$

$$\therefore \Omega_p^* = \frac{2}{0.0167} \tan\left(\frac{20\pi \times 0.0167}{2}\right) \quad \text{...Here } \omega = 20\pi \text{ (given)}$$

$$\therefore \Omega_p^* = 69.31$$

Now the value of $H^*(s)$ is obtained by putting $s = \frac{s}{\Omega_p^*}$ in Equation (1).

$$\therefore H^*(S) = \frac{1}{\frac{s}{\Omega_p^*} + 2}$$

$$\therefore H^*(S) = \frac{1}{\frac{s}{69.31} + 2}$$

$$\therefore H^*(S) = \frac{69.31}{s + 138.62} \quad ...(2)$$

Now H (Z) is obtained by putting $s = \frac{2}{T_s}\left(\frac{Z-1}{Z+1}\right)$ in Equation (2),

$$\therefore H(Z) = \frac{69.31}{\frac{2}{0.0167}\left(\frac{Z-1}{Z+1}\right) + 138.62} = \frac{69.31}{119.76\left(\frac{Z-1}{Z+1}\right) + 138.62}$$

$$\therefore H(Z) = \frac{69.31(Z+1)}{119.76\,Z - 119.76 + 138.62\,Z + 138.62}$$

$$\therefore H(Z) = \frac{69.31(Z+1)}{258.38\,Z + 18.86}$$

$$\therefore H(Z) = \frac{Z+1}{3.73\,Z + 0.27} \qquad \therefore H(Z) = \frac{1+Z^{-1}}{3.73 + 0.27\,Z^{-1}}$$

This is the required transfer function for digital filter.

1.2.4 Matched Z transform :

In this method, the analog filter is transformed into the digital filter by mapping poles and zeros of H (s) directly into the poles and zeros in Z plane.

Consider that the transfer function of analog filter is given by,

$$H(s) = \frac{\sum_{k=1}^{M} (s - Z_k)}{\sum_{k=1}^{M} (s - P_k)} \quad ...(1)$$

Here Z_k represents the zeros of system and P_k represents the poles of system.

In this method; the poles obtained from matched Z transform are identical to poles obtained with impulse invariance method. But the zero positions are different than impulse invariance method.

The transfer function of digital filter can be given by,

$$H(Z) = \frac{\sum_{k=1}^{M} (1 - e^{Z_k T_s} \cdot Z^{-1})}{\sum_{k=1}^{M} (1 - e^{P_k T_s} \cdot Z^{-1})}$$

Here T_s is the sampling time. In this method, to maintain the frequency response characteristics of analog filter normal; the sampling interval (T_s) is kept very low in order to avoid aliasing.

1.2.5 Least squares filter design :

This method is used to design digital IIR filter directly without doing any conversion. Consider that we have to design digital IIR filter which contains only poles. The transfer function of such filter is given by,

$$H(Z) = \frac{b_0}{1 + \sum_{k=1}^{N} a_k Z^{-k}} \quad ...(1)$$

The desired transfer function is denoted by $H_d(Z)$. Consider the cascade connection of $H_d(Z)$ with $\frac{1}{H(Z)}$ as shown in Fig. A. Note that $\frac{1}{H(Z)}$ is all zero system.

Consider that input applied to this cascade connection is $\delta(n)$. Actually $y(n) = \delta(n)$; this is because we are cascading a filtering and inverse filtering operation. Here H (Z) is approximation of desired frequency response $H_d(Z)$.

As shown in Fig.K-4 the input applied to inverse filter is $h_d(n)$ and its response is $\frac{1}{H(Z)}$.

From Equation (1) we have,

$$\frac{1}{H(Z)} = \frac{1 + \sum_{k=1}^{N} a_k Z^{-k}}{b_0} \qquad ...(2)$$

The actual output is,

$$y(n) = \frac{1}{b_0}\left[h_d(n) + \sum_{k=1}^{N} a_k h_d(n-k) \right] \qquad ...(3)$$

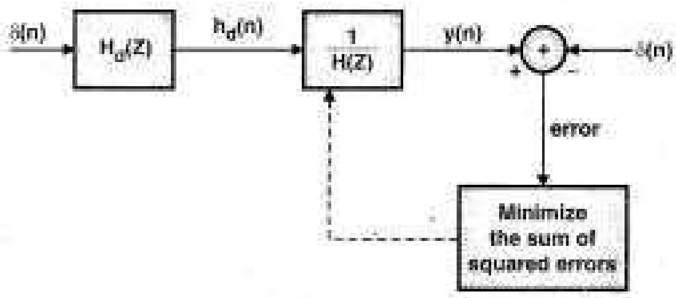

Fig.K-4 : Least square inverse filter design

Here for $n > 0$, $y(n)$ represents the error between the desired output and the actual output. Here the parameter a_k is selected to minimize the sum of squares of error sequence.

$$\varepsilon = \sum_{n=1}^{\infty} y^2(n) \qquad ...(4)$$

Putting Equation (3) in Equation (4) we get,

$$\varepsilon = \sum_{n=1}^{\infty} \left\{ \frac{1}{b_0}\left[h_d(n) + \sum_{k=1}^{N} a_k h_d(n-k) \right] \right\}^2 \qquad ...(5)$$

By differentiating Equation (5) with respect to a_k results into the set of linear differential equations. By solving these equations the coefficients of desired filter can be obtained.